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Measurements of Entanglement and a Quantum de Finetti Theorem

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The problem of defining a natural measure of entanglement of mixed states on tensor products is considered from the point of view of a quantum de Finetti theorem for Bosons.

KEY WORDS: entanglement; Boson de Finetti theorem.

1. INTRODUCTION

Consider a Hilbert space tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ providing the quantum theoretical description of a bipartite composite physical system $S = S_1 \times S_2$. A pure state of S is represented by a unit vector $\psi \in \mathcal{H}$. If ψ is a product vector, $\psi = \psi_1 \otimes \psi_2$, the state is said to be *unentangled*; if not it is *entangled*. Representing the state alternatively by a density operator (that is, nonnegative operator of unit trace on $\mathcal{H}$) ρ, it is evident that a necessary condition for unentanglement is that ρ is the tensor product operator

$$\rho = \rho_1 \otimes \rho_2 \tag{1}$$

of density operators ρ_1 on $\mathcal{H}_1$ and ρ_2 on $\mathcal{H}_2$, respectively. Indeed, since $\rho = |\psi\rangle\langle\psi|$ is the projector onto the one-dimensional subspace spanned by ψ we have $\rho_1 = |\psi_1\rangle\langle\psi_1|$ and $\rho_2 = |\psi_2\rangle\langle\psi_2|$. Conversely if a pure state density operator ρ satisfies the factorization condition (1) then, since for any density operator σ, $\operatorname{tr}\sigma^2 \leq 1$ with equality if and only if the corresponding state is pure, we have

$$1 = \operatorname{tr}\rho^2 = \operatorname{tr}(\rho_1 \otimes \rho_2)^2 = \operatorname{tr}\left(\rho_1^2 \otimes \rho_2^2\right) = \operatorname{tr}\rho_1^2 \operatorname{tr}\rho_2^2 \leq 1,$$

so that it follows that ρ_1 and ρ_2 must represent pure states. Hence (1) is a necessary and sufficient condition for a pure state to be unentangled; we take it as the definition of unentanglement also for mixed states.

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Consider a possibly entangled pure state ω represented by the unit vector ψ and by the density operator $\rho = |\psi\rangle\langle\psi|$. It is generally agreed that an appropriate measure of the degree of entanglement of ω is given by

$$\varepsilon(\omega) = S\{\mathrm{tr}_{\mathcal{H}_2}[\rho]\}.$$

Here $\mathrm{tr}_{\mathcal{H}_2}[\rho]$ is the partial trace of the density operator ρ over the Hilbert space $\mathcal{H}_2$, that is the unique density operator ρ_1 on $\mathcal{H}_1$ such that, for every bounded operator T on $\mathcal{H}_1$,

$$\mathrm{tr}(\rho_1 T) = \mathrm{tr}\{\rho(T \otimes 1_{\mathcal{H}_2})\}$$

where the trace on the left is over $\mathcal{H}_1$ and that on the right is over $\mathcal{H}$, and $S\{\sigma\}$ denotes the von Neumann entropy

$$S\{\sigma\} = -\mathrm{tr}\{\sigma \ln \sigma\}$$

of the density operator σ. For example $\varepsilon(\omega)$ vanishes if ω is unentangled and attains its maximum value $\ln d$ (in the case when $\dim \mathcal{H}_1 = d$ is finite and $\dim \mathcal{H}_2 \geq d$) where the unit vector ψ has the maximally entangled form

$$\psi = d^{-1} \sum_{j=1}^{d} \phi_j \otimes \chi_j$$

where the ϕ_j are orthonormal vectors in $\mathcal{H}_1$ and the χ_j are orthonormal vectors in $\mathcal{H}_2$. More generally, if ψ is given in terms of such orthonormal sets by

$$\psi = \sum_{j=1}^{d} \lambda_j \phi_j \otimes \chi_j$$

where the λ_j are nonnegative numbers whose sum is 1, then

$$\varepsilon(\omega) = -\sum_{j=1}^{d} \lambda_j \ln \lambda_j.$$

It is evident from this formula that $\varepsilon(\omega)$ could be defined equally well by

$$\varepsilon(\omega) = S\{\mathrm{tr}_{\mathcal{H}_1}[\rho]\}.$$

No such agreement attends the definition of a measure of entanglement for mixed states. The natural definition, that for a mixed state ω which is a convex combination of pure states,

$$\omega = \sum_{j=1}^{N} \lambda_j \omega_j,$$

$$\varepsilon(\omega) = \sum_{j=1}^{N} \lambda_j \varepsilon(\omega_j), \tag{2}$$

fails because the decomposition into a convex combination of pure states is nonunique. Majewski (2001) and others have proposed to define $\varepsilon(\omega)$ as the infimum of the right-hand side of (2) over all such decompositions of ω. Our purpose in this note is to advocate a change of point of view to this question, based on a certain quantum de Finetti theorem (Hudson and Moody, 1976) which, as already noted (Hudson, 1981) has a bearing on the question of the nonuniqueness of the resolution of a mixed state into a convex combination of pure states. The use of the associated notion of indistinguishability in physics has been strongly advocated by Bach (1997).

2. CLASSICAL AND QUANTUM DE FINETTI THEOREMS

Following de Finetti (1937), a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ on a common classical probability space is called *indistinguishable* if for all $n \in \mathbb{N}$, all choices of distinct $j_1, j_2, \ldots, j_n$, $n \in \mathbb{N}$ and all choices of permutation π of $\{1, 2, \ldots, n\}$ the joint probability distribution $\mathbb{P}_{j_1, j_2, \ldots, j_n}$ of $X_{j_1}, X_{j_2}, \ldots, X_{j_n}$ is identical to $\mathbb{P}_{j_{\pi(1)}, j_{\pi(2)}, \ldots, j_{\pi(n)}}$. Thus an exchangeable sequence is characterized by a sequencence of probability distributions $(P_n)_{n\in\mathbb{N}}$, where $P_n = \mathbb{P}_{1,2,\ldots,n}$ is a distribution on $\mathbb{R}^n$ satisfying the following conditions:

- Each P_n is *symmetric*, that is, for each permutation π we have $P_n = P_n \circ \pi$ where π acts on $\mathbb{R}^n$ by permuting coordinates.
- The sequence $(P_n)_{n\in\mathbb{N}}$ is *consistent*, that is for each $n \in \mathbb{N}$ and each Borel subset A of $\mathbb{R}^n$ we have

$$P_n(A) = P_{n+1}(A \times \mathbb{R}).$$

Conversely using the Kolmogorov consistency theorem, it is easily seen that every consistent symmetric sequence of probability measures defines a corresponding sequence of exchangeable random variables. De Finetti's famous theorem [de Finetti, 1937] can be stated in the language of such sequences as follows.

Theorem 1. *Let there be given a symmetric consistent sequence of probability distributions $(P_n)_{n\in\mathbb{N}}$. Then there exists a unique probability measure $\mathbb{P}$ on the σ-field $\mathcal{F}$ of subsets of the set $\mathcal{P}_1$ of probability distributions on $\mathbb{R}$ generated by the functions*

$$\mathcal{P}_1 \ni P \mapsto P(A)$$

where A is a Borel subset of $\mathbb{R}$, such that, for arbitrary $n \in \mathbb{N}$,

$$P_n = \int_{P \in \mathcal{P}_1} (\otimes^n P)\ \mathbb{P}(\mathrm{dP}),$$

where $\otimes^n P$ *denotes the product probability measure on* $\mathbb{R}^n$ *of* n *copies of* P.

A quantum analogue of this theorem (Hudson and Moody, 1976) is easily formulated. Let there be given a Hilbert space $\mathcal{H}$ and, for each $n \in \mathbb{N}$, a density operator ρ_n on the tensor product $\otimes^n \mathcal{H}$ satisfying the following conditions:

- *symmetry*: for each permutation π of $\{1, 2, \ldots, n\}$,

$$U_\pi \rho_n U_\pi^{-1} = \rho_n \tag{3}$$

  where U_π denotes the unitary action of the permutation on $\otimes^n \mathcal{H}$ which permutes the components of product tensors.
- *consistency*: for each $n \in \mathbb{N}$ and each bounded operator T on $\otimes^n \mathcal{H}$ we have

$$\mathrm{tr}\, \rho_n T = \mathrm{tr}\{\rho_{n+1}(T \otimes 1_{\mathcal{H}})\}$$

  where the trace on the left is over $\otimes^n \mathcal{H}$ while that on the right is over $\otimes^{n+1} \mathcal{H}$.

Denote by $\mathcal{D}$ the set of density operators on $\mathcal{H}$ and by $\mathcal{F}$ the σ-field of subsets of $\mathcal{D}$ generated by the functions

$$\mathcal{D} \ni \rho \mapsto \mathrm{tr}\, \rho T$$

where T is a bounded operator on $\mathcal{H}$. Then the natural analog of Theorem 1 is the following:

Theorem 2. *Let there be given a symmetric consistent sequence of density operators* $(\rho_n)_{n \in \mathbb{N}}$. *Then there exists a unique probability measure* $\mathbb{P}$ *on* $(\mathcal{D}, \mathcal{F})$ *such that for each* $n \in \mathbb{N}$

$$\rho_n = \int_{\rho \in \mathcal{D}} (\otimes^n \rho) \mathbb{P}(d\rho).$$

For a proof of Theorem 2 see (Hudson and Moody, 1976). A closely related but more abstract C^*-algebraic version of this theorem was proved earlier by Stϕrmer (1969).

We now describe a corollary of Theorem 2 which has no classical analogue. We say that a density operator ρ_n on $\otimes^n \mathcal{H}$ is *Bose-symmetric* if instead of (3) above it satisfies

$$U_\pi \rho_n = \rho_n$$

for every permutation π of $\{1, 2, \ldots, n\}$. Equivalently ρ_n is supporeted by the symmetric sector

$$\otimes^n_{\text{sym}}\mathcal{H} = \left((n!)^{-1}\sum_{\pi} U_\pi\right)(\otimes^n\mathcal{H})$$

of the tensor product $\otimes^n\mathcal{H}$. Every Bose-symmetric density operator is symmetric in the sense of (3).

We denote by $\mathcal{E}(\mathcal{D})$ the set of extreme points of the convex set $\mathcal{D}$; equivalently $\mathcal{E}(\mathcal{D})$ is the set of one-dimensional projectors

$$\mathcal{E}(\mathcal{D}) = \{|\psi\rangle\langle\psi| : \psi \in \mathcal{H}, \|\psi\| = 1\}.$$

Theorem 3. *Let the consistent sequence of density operators $(\rho_n)_{n\in\mathbb{N}}$ be Bose-symmetric. Then the probability measure $\mathbb{P}$ of Theorem 2 is supported by $\mathcal{E}(\mathcal{D})$.*

Thus every Bose-symmetric consistent sequence $(\rho_n)_{n\in\mathbb{N}}$ is uniquely expressible in the form

$$\rho_n = \int_{\rho\in\mathcal{E}(\mathcal{D})} (\otimes^n\rho)\mathbb{P}(d\rho). \tag{4}$$

3. MEASURES OF ENTANGLEMENT OF MIXED STATES

We consider again the Hilbert space tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ describing a composite quantum system. For each $n \in \mathbb{N}$ there is a natural isomorphism

$$\otimes^n\mathcal{H} \equiv (\otimes^n\mathcal{H}_1) \otimes (\otimes^n\mathcal{H}_2)$$

which permutes the order of product vectors appropriately, which we use to identify these Hilbert spaces. Given an unentangled pure state described by the product unit vector $\psi = \psi_1 \otimes \psi_2$ we evidently have

$$\otimes^n\psi = (\otimes^n\psi_1) \otimes (\otimes^n\psi_2)$$

so that the corresponding multiparticle state described by the unit vector $\otimes^n\psi$ is also unentangled. Now consider an entangled pure state ω with corresponding unit vector ψ, so that

$$\varepsilon(\omega) = S\{\mathrm{tr}_{\mathcal{H}_2}[\rho]\}$$

where $\rho = |\psi\rangle\langle\psi|$. For the pure multiparticle product state $\otimes^n\omega$ with unit vector $\otimes^n\psi$ we have

$$\begin{aligned}\varepsilon(\otimes^n\omega) &= S\{\mathrm{tr}_{\otimes^n\mathcal{H}_2}[\otimes^n\rho]\}\\ &= S\{\otimes^n(\mathrm{tr}_{\mathcal{H}_2}[\rho])\}\end{aligned}$$

$$= \sum_{j=1}^{n} S\{\mathrm{tr}_{\mathcal{H}_2}[\rho]\}$$
$$= n\varepsilon(\omega)$$

where we used the well-known identity

$$S\{\rho_1 \otimes \rho_2\} = S\{\rho_1\} + S\{\rho_2\}.$$

Thus the *entanglement measure per particle* $n^{-1}\varepsilon(\otimes^n \omega)$ of the multiparticle state is independent of n. This seems to be an intuitively well-founded principle. Note that it is implicit in this principle that the particles are Bosons, so that $\otimes^n \psi$ describes an allowable multiparticle state.

Finally let us consider mixed states. We wish to define the entanglement measure in such a way as to respect this principle. We immediately have to face the problem that in general the product state $\otimes^n \omega$ of a mixed one-particle state ω violates Boson statistics. For example if $\mathcal{H}$ is two dimensional, with orthonormal basis (ψ_1, ψ_2), and ω is described by the density operator $\lambda|\psi_1\rangle\langle\psi_1| + (1-\lambda)|\psi_2\rangle\langle\psi_2|$ with $0 < \lambda < \frac{1}{2}$ then $\otimes^2\omega$ is described by the density operator on $\otimes^2\mathcal{H}$

$$\begin{aligned}\rho \;&= \lambda^2|\psi_1 \otimes \psi_1\rangle\langle\psi_1 \otimes \psi_1| + (1-\lambda)^2|\psi_2 \otimes \psi_2\rangle\langle\psi_2 \otimes \psi_2| \\ &\quad + \lambda(1-\lambda)(|\psi_1 \otimes \psi_2\rangle\langle\psi_1 \otimes \psi_2| + |\psi_2 \otimes \psi_1\rangle\langle\psi_2 \otimes \psi_1|)\end{aligned}$$

and it is evident that $U_{(1,2)}\rho \neq \rho$ even though $U_{(1,2)}\rho U_{(1,2)}^{-1} = \rho$. Thus, assuming Boson statistics, there is no canonical way of embedding the one-particle state ω in a hierarchy of multiparticle states. Instead we must consider that *all hierarchies based on the given one particle state which satisfy Boson statistics and the consistency principle are equally valid*. A consequence is that it is not possible to define a canonical measure of entanglement for a mixed single-particle state. Instead we expect to be able to define such a measure only for a given embedding in a Boson-symmetric consistent hierarchy $(\omega_n)_{n\in\mathbb{N}}$.

Theorem 3 tells us how to do this. Representing each state ω_n by the density operator ρ_n, by Theorem 3 we can represent the sequence $(\rho_n)_{n\in\mathbb{N}}$ in the form (4). The entanglement measure per unit particle of each multiparticle state can then be defined by

$$n^{-1}\varepsilon(\omega_n) = n^{-1}\int_{\rho\in\mathcal{E}(\mathcal{D})} \varepsilon(\omega_{\otimes^n\rho})\mathbb{P}(d\rho)$$

where $\omega_{\otimes^n\rho}$ is the pure multiparticle state corresponding to the density operator $\otimes^n\rho$. In particular the entanglement measure of the one-particle state ω is given in terms of pure state measures of entanglement by

$$\varepsilon(\omega) = \int_{\rho\in\mathcal{E}(\mathcal{D})} \varepsilon(\omega_\rho)\mathbb{P}(d\rho).$$

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